

Cosine operator and controllability of the wave equation with memory revisited

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July 31, 2014

1 Introduction

In this paper we consider the following equation with memory:

$$w' = 2\alpha w(t) + \int_0^t N(t-s)\Delta w(s) \, ds \quad (1)$$

where $w = w(x, t)$ with $t > 0$ and $x \in \Omega$, a region with C^2 boundary and $N \in H^3(0, T)$ for every $T > 0$. We associate the initial and boundary conditions

$$\begin{cases} w(0) = w_0 \in L^2(\Omega), \\ w(t) = f(t) \text{ if } x \in \Gamma \subseteq \Omega, \quad w(t) = 0 \text{ if } x \in \partial\Omega \setminus \Gamma \end{cases}$$

(Γ is relatively open in $\partial\Omega$. The case $\Gamma = \partial\Omega$ is not excluded).

The function f is a control which we use to steer the initial datum $w_0 \in L^2(\Omega)$ to a target $\xi \in L^2(\Omega)$ at a certain time T . This kind of control problem has been studied by several authors and with different methods, since a system of the form (1) is important for the applications in viscoelasticity, thermodynamics of materials with memory and nonfickian diffusion. Note that in viscoelasticity also controllability of the pair (w, w') of the deformation and velocity has to be studied but here for simplicity we confine ourselves to the controllability of the sole component w . It is a fact that: 1) the controllability of the sole component w is sufficient for the solutions of source identification problems, see [11, 9, 12]; 2) also the controllability of the pair of the deformation and the stress (or the flux) has its interest, and this is a new problem which appears in the case of systems with memory, see [1, 2, 13].

The key idea which underlines essentially all the papers on controllability of Eq. (1) is that the controllability properties of the *associated wave equation*

$$u'' = \Delta u + F, \quad \begin{cases} u(0) = u_0, \quad u'(0) = u_1, \\ u(t) = f(t) \text{ if } x \in \Gamma \subseteq \Omega, \quad u(t) = 0 \text{ if } x \in \partial\Omega \setminus \Gamma \end{cases} \quad (2)$$

can be lifted to the system with memory (1).

The paper [10] proved that the control properties of (2) can be lifted to the system (1) using cosine operator theory. Here we intend to revise and improve this approach.

The organization of the paper is as follows: first we combine the MacCamy trick to give a definition/representation of the solutions of Eq. (1), see section 3. In particular we prove that for every $f \in L^2(0, T; L^2(\Gamma))$ and every $w_0 \in L^2(\Omega)$ a solution exists, such that $w \in C([0, T]; L^2(\Omega))$ for every $T > 0$. This justify the following definition of the reachable set at time T (the index M is for “memory”):

$$R_M(T) = \{w(T), \quad f \in L^2(0, T; L^2(\Gamma))\}.$$

The final result is the proof that if the wave equation is controllable then system (1) is controllable too (see the precise statement in Theorem 6). This is in two steps: in the first step we prove that $R^\perp(T)$ is finite dimensional and then we prove that its orthogonal is reduced to the subspace 0.

These arguments depend on known properties of the wave equation, which are recalled in Section 2.

1.1 Comments on previous results

Controllability of Equation (1) has been studied using different methods which are reviewed in [15]. The papers [5] uses Fourier expansions and moment methods, an approach extended in [14] (see also [15] and references therein). Extension to (1) of the inverse inequality of the wave equation is in [8] while Carlemm estimates are used in [3]. Here we extend and improve the operator approach in [10].

2 The properties of the wave equation

We need few pieces of information on the wave equation (2). We introduce the operators A , \mathcal{A} and D :

$$\mathcal{A} = i(-A)^{1/2} \quad \text{where} \quad \text{dom } A = H^2(\Omega) \cap H_0^1(\Omega), \quad A\phi = \Delta\phi$$

while the operator D , the Dirichlet operator, is defined by

$$u = Df \iff \Delta u = 0 \quad u(x) = f(x) \text{ on } \Gamma, \quad u = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

The operator \mathcal{A} generates a strongly continuous group, so that we can consider the strongly continuous operators $R_+(t)$ and $R_-(t)$ defined by:

$$R_+(t) = \frac{1}{2} [e^{\mathcal{A}t} + e^{-\mathcal{A}t}] , \quad R_-(t) = \frac{1}{2} [e^{\mathcal{A}t} - e^{-\mathcal{A}t}] .$$

The operator $R_+(t)$ is the *cosine operator* generated by A and its key property is

$$R_+(t)R_+(\tau) = \frac{1}{2} [R_+(T + \tau) + R_+(t - \tau)] .$$

This equality holds for every real t and τ .

Let $u_0 \in L^2(\Omega)$, $u_1 \in H^{-1}(\Omega)$, $F \in L^1(0, T; L^2(\Omega))$ and $f \in L^2(0, T; L^2(\Gamma))$. It is known that problem (2) admits a unique solution $u \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ which is given by

$$\begin{aligned} u(t) &= R_+(t)u_0 + \mathcal{A}^{-1}R_-(t)u_1 + \mathcal{A}^{-1} \int_0^t R_-(t-s)F(s) \, ds - \\ &\quad - \mathcal{A} \int_0^t R_-(t-s)Df(s) \, ds, \\ u'(t) &= \mathcal{A}R_-(t)u_0 + R_+(t)u_1 + \int_0^t R_+(t-s)F(s) \, ds - \mathcal{A} \int_0^t R_+(t-s)Df(s) \, ds. \end{aligned} \quad (3)$$

The following result is known (see [6]). Let γ_1 be the exterior normal derivative,

$$\gamma_1 \phi(x) = \frac{\partial}{\partial n} \phi(x), \quad x \in \partial\Omega.$$

Theorem 1 *The following properties hold for the memoryless wave equation (2). We state separately the effects of u_0 , u_1 , F and of the boundary control f .*

1. *Let $f = 0$ and $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $F \in L^1(0, T; L^2(\Omega))$. Then $u(t) \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ and it is a linear and continuous function of u_0 , u_1 , F in the specified spaces. Furthermore, for every $T > 0$ there exists $M > 0$ such that*

$$\int_{\Gamma} \int_0^T |\gamma_1 u(t)|^2 \, dt \, d\Gamma \leq M \left[\|u_0\|_{H_0^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 + \|F\|_{L^1(0, T; L^2(\Omega))}^2 \right]. \quad (4)$$

2. *If $f = 0$ and $u_0 \in L^2(\Omega)$, $u_1 \in H^{-1}(\Omega)$, $F \in L^1(0, T; L^2(\Omega))$ then $u(t) \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ and it is a linear and continuous function of u_0 , u_1 , F in the specified spaces.*
3. *if $f \in L^2(0, T; L^2(\Gamma))$ and $u_0 = 0$, $u_1 = 0$, $F = 0$ then $u(t) \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ and depends continuously on f .*

The previous properties justify the following definition, where the control time is called $2T$ for later convenience:

Definition 2 *The wave equation (2) is controllable at time $2T$ if for every u_0 and ξ in $L^2(\Omega)$, u_1 and η in $H^{-1}(\Omega)$ and $F \in L^1(0, T; L^2(\Omega))$ there exists a control $f \in L^2(0, T; L^2(\Gamma))$ such that*

$$u(2T) = \xi, \quad u'(2T) = \eta.$$

It is known that

1. if Γ is “too small” then there exists no time at which the wave equation is controllable.

2. there exist subset Γ of $\partial\Omega$ (for example, $\Gamma = \partial\Omega$) such that controllability holds for a suitable time.
3. if controllability holds at time $2T$ then it holds also at every larger time.
4. controllability does not depend on u_0 , u_1 and F so that when studying controllability we can assume $u_0 = u_1 = 0$, $F = 0$. So, controllability is the property that the following map is surjective. The map acts from $L^2(0, T; L^2(\Gamma))$ to $L^2(\Omega) \times H^{-1}(\Omega)$ and it is defined by

$$\begin{aligned} f &\mapsto \Lambda_0(2T)f = (\Lambda_0^1(2T), \Lambda_0^2(2T)) f = \\ &= \left(\mathcal{A} \int_0^{2T} R_-(2T-s) Df(s) \, ds, \mathcal{A} \int_0^{2T} R_+(2T-s) Df(s) \, ds \right). \end{aligned}$$

2.1 A consequence in terms of bases

It is known that the operator A is selfadjoint with compact resolvent. Hence, $L^2(\Omega)$ has an orthonormal basis whose elements $\phi_n(x)$ are eigenvectors of A :

$$A\phi_n = -\lambda_n^2 \phi_n.$$

It is a fact that $\lambda_n^2 > 0$ hence λ_n is real and we can choose $\lambda_n > 0$. The eigenvalues are not distinct, but the eigenvectors with the same eigenvalue are finite in number.

The operators $R_+(t)$ and $R_-(t)$ have a simple representation in terms of $\phi_n(x)$:

$$\begin{aligned} R_+(t) \left(\sum_{n=1}^{+\infty} c_n \phi_n(x) \right) &= \sum_{n=1}^{+\infty} \phi_n(x) (c_n \cos \lambda_n t), \\ R_-(t) \left(\sum_{n=1}^{+\infty} c_n \phi_n(x) \right) &= \sum_{n=1}^{+\infty} \phi_n(x) (c_n \sin \lambda_n t). \end{aligned}$$

Furthermore, we know that (see [16, Prop. 10.6.1] and note that our operator A is $-A_0$ in [16])

$$\int_{\Omega} \phi_n(x) Df \, dx = -\frac{1}{\lambda_n^2} \int_{\Gamma} (\gamma_1 \phi_n) f \, d\Gamma.$$

So, $-\Lambda_0(2T)f$ has the following “concrete” representation:

$$\begin{aligned} &\left(\sum_{n=1}^{+\infty} \phi_n(x) \int_0^{2T} \int_{\Gamma} \left(\frac{\gamma_1 \phi_n}{\lambda_n} \right) (\sin \lambda_n s) f(x, 2T-s) \, d\Gamma \, ds, \right. \\ &\left. \sum_{n=1}^{+\infty} (\lambda_n \phi_n(x)) \int_0^{2T} \int_{\Gamma} \left(\frac{\gamma_1 \phi_n}{\lambda_n} \right) (\cos \lambda_n s) f(x, 2T-s) \, d\Gamma \, ds \right). \end{aligned}$$

It is known that $\{\lambda_n \phi_n\}$ is an orthonormal basis of $H^{-1}(\Omega)$ (in an inner product whose norm is equivalent to the standard norm) so that every target has the representation

$$\xi = \sum_{n=1}^{+\infty} \xi_n \phi_n, \quad \eta = \sum_{n=1}^{+\infty} \eta_n (\lambda_n \phi_n) \quad \{\xi_n\} \in l^2, \quad \{\eta_n\} \in l^2.$$

Controllability is equivalent to the solvability of the following moment problem, in terms of a *real* function f :

$$\int_0^{2T} \int_{\Gamma} \Psi_n e^{i\lambda_n s} f(x, 2T-s) \, d\Gamma \, ds = \eta_n + i\xi_n = c_n, \quad n \in \mathbb{N} \quad \Psi_n = \frac{\gamma_1 \phi_n}{\lambda_n}. \quad (5)$$

Note that $\{c_n\}$ is an *arbitrary complex valued l^2 -sequence*.

We introduce

$$\mathbb{Z}' = \mathbb{Z} \setminus \{0\}, \quad \lambda_n = -\lambda_{-n}, \quad \phi_n = \phi_{-n} \text{ for } n < 0.$$

Then, the moment problem (5) is equivalent to

$$\int_0^{2T} \int_{\Gamma} \Psi_n e^{i\lambda_n s} h(x, 2T-s) \, d\Gamma \, ds = c_n, \quad n \in \mathbb{Z}' \quad (6)$$

where now $h \in L^2(0, 2T; L^2(\Gamma))$ is *complex valued* and $\{c_n\} \in l^2(\mathbb{Z}')$ is arbitrary.

We introduce the moment operator

$$\mathbb{M}_0 h = \int_0^{2T} \int_{\Gamma} \Psi_n e^{i\lambda_n s} h(x, 2T-s) \, d\Gamma \, ds.$$

The fact that the transformation $f \mapsto \Lambda_0(2T)f$ is continuous from $L^2(0, 2T; L^2(\Gamma))$ to $L^2(\Omega) \times H^{-1}(\Omega)$ implies that $\mathbb{M}_0 \in \mathcal{L}(L^2(0, 2T; L^2(\Gamma)), l^2(\mathbb{Z}'))$. So, its domain is $L^2(0, 2T; L^2(\Gamma))$ and its restriction to $\text{clspan}\{\Psi_n e^{i\lambda_n t}\}$ is invertible.

If the wave equation is controllable at time $2T$ then \mathbb{M}_0 is *surjective*, so that its inverse (as an operator from $\text{clspan}\{\Psi_n e^{i\lambda_n t}\} \subseteq L^2(0, 2T; L^2(\Gamma))$ to $l^2(\mathbb{Z}')$) is bounded. This implies (see [7, p. 22] and [14, 15]):

Theorem 3 *Let the associated wave equation be controllable in time $2T$. Then:*

- *the sequence $\{\Psi_n e^{i\lambda_n t}\}_{n \in \mathbb{Z}'}$ is a Riesz sequence in $L^2(0, 2T; L^2(\Gamma))$;*
- *the sequences $\{\Psi_n \cos \lambda_n t\}_{n \in \mathbb{N}}$, $\{\Psi_n \sin \lambda_n t\}_{n \in \mathbb{N}}$ are Riesz sequences in $L^2(0, T; L^2(\Gamma))$;*
- *the operator $\Lambda_0^1(T)$ is surjective.*

3 The solutions of System (1)

Different methods have been proposed to study the solutions of Eq. (1). Here we follow a method based on the use of cosine operator theory.

We apply a transformation, known as MacCamy trick, to the solutions of Eq. (1). This is a formal step, since the solutions are not yet defined. Formally, computing the derivatives of both the sides of (1) we get

$$w'' = 2\alpha w' = \left[\Delta w + \int_0^t N'(t-s) \Delta w(s) \, ds \right] + F(t) \quad (7)$$

(the affine term should be zero. We inserted it here since even if $F = 0$ an affine term will appear in the following computations).

We consider (7) as a Volterra integral equation in the unknown Δw . Let $\tilde{M}(t)$ be the resolvent kernel of $M(t) = N'(t)$, i.e. $\tilde{M}(t)$ is the unique solution of

$$\tilde{M}(t) = M(t) - \int_0^t N'(t-s) \tilde{M}(s) \, ds.$$

Then we get formally

$$\Delta w(t) = w''(t) - 2\alpha w'(t) - F - \int_0^t \tilde{M}(t-s) [w''(s) - 2\alpha w'(s) - F(s)] \, ds.$$

In this equation, $w(0) = w_0$ and $w'(0) = 0$. For the following it is convenient to study this equation in the more general case

$$w'(0) = w_1,$$

possibly different from zero. We integrate by parts and we find a system of the following form:

$$w''(t) = \Delta w(t) + aw'(t) + bw(t) + \int_0^t \tilde{M}_1(t-s)w(s) \, ds + F_1(t) \quad (8)$$

where a and b are suitable constants and

$$F_1(t) = F(t) - \int_0^t \tilde{M}(t-s)F(s) \, ds - \tilde{M}(t)w_1 - \tilde{M}'(t)w_0.$$

Note the dependence of $F_1(t)$ on the initial conditions w_0 and w_1 and note also that after the MacCamy trick the laplacian does not appear in the memory of the integral.

It is simple to see that the transformation $w(t) \mapsto e^{-at/2}w(t)$ can be used to remove the velocity term from (8) (this changes $\tilde{M}(t)$ to $M(t) = e^{-2at/2}\tilde{M}(t) = K(t)$ and similar transformation of $F_1(t)$ and the boundary control f). So, we study the problem

$$\begin{aligned} w''(t) &= \Delta w(t) + bw(t) + \int_0^t K(t-s)w(s) \, ds + F(t), \\ w(0) &= w_0, \quad w'(0) = w_1 \\ w &= f \text{ on } \Gamma, \quad w = 0 \text{ on } \partial\Omega \end{aligned} \quad (9)$$

where $F_1(t)$ has been renamed $F(t)$ and it is a continuous affine function of w_0 and w_1 .

This is a perturbed wave equation and we can use formula (3) in order to get a Volterra integral equation for $w(t)$. It is convenient to write separately the formula for the contribution of the boundary control f and for the contribution of w_0 , w_1 and F :

$$\begin{aligned} w(t) = & R_+(t)w_0 + \mathcal{A}^{-1}R_-(t)w_1 + \mathcal{A}^{-1} \int_0^t R_-(t-s)F(s) \, ds + \\ & + \mathcal{A}^{-1} \int_0^t R_-(t-s) \left[bw(s) + \int_0^s K(s-r)w(r) \, dr \right] \, ds, \quad (10) \end{aligned}$$

$$\begin{aligned} w(t) = & -\mathcal{A} \int_0^t R_-(t-s)Df(s) \, ds + \\ & + \mathcal{A}^{-1} \int_0^t R_-(t-s) \left[bw(s) + \int_0^s K(s-r)w(r) \, dr \right] \, ds. \quad (11) \end{aligned}$$

The general solutions is the sum of the two but for our applications we keep distinct the two formulas.

Both the formulas (10) and (11) for $w(t)$ have the following general form

$$w(t) = u(t) + \mathcal{A}^{-1} \int_0^t L(t-s)w(s) \, ds, \quad L(t)w = bR_-(t)w + \int_0^t K(t-r)R_-(r)w \, dr. \quad (12)$$

Note that $u(t)$ solves the associated wave equation.

This is a Volterra integral equation for $w(t)$ which we solve using Picard iteration:

$$\begin{aligned} w(t) &= u(t) + \mathcal{A}^{-1} \int_0^t L(t-s)w(s) \, ds = \\ &= u(t) + \mathcal{A}^{-1} \int_0^t L(t-s)u(s) \, ds + \sum_{k=2}^{+\infty} (\mathcal{A}^{-1})^k L^{*k} * u \quad (13) \end{aligned}$$

where the exponent *k denotes iterated convolution.

We introduce the kernel $H(t)$:

$$H(t) = \sum_{k=1}^{+\infty} (\mathcal{A}^{-1})^k (L)^{*k} = \mathcal{A}^{-1} \left(\sum_{k=1}^{+\infty} (\mathcal{A}^{-1})^{k-1} (L^{*k}) \right)$$

so that

$$w(t) = u(t) + \int_0^t H(t-s)u(s) \, ds.$$

When the boundary control is $f = 0$ this formula specializes to

$$w(t) = R_+(t)w_0 + \mathcal{A}^{-1}R_-(t)w_1 + \mathcal{A}^{-1} \int_0^t R_-(t-s)F(s) \, ds + \int_0^t H(t-s) [R_+(s)w_0 + \mathcal{A}^{-1}R_-(s)w_1 + \mathcal{A}^{-1} \int_0^s R_-(s-r)F(r) \, dr] \, ds \quad (14)$$

while the corresponding formula with $w_0 = w_1 = 0$ and $F = 0$ is

$$w(t) = -\mathcal{A} \int_0^T R_-(t-s)Df(s) \, ds - \int_0^t H(t-s)\mathcal{A} \int_0^s R_-(s-r)Df(r) \, dr \, ds \quad (15)$$

The properties of the solutions of the wave equation that we recalled in Sect. 2 imply:

Theorem 4 *Let $F \in L^1(0, T; L^2(\Omega))$, $f \in L^2(0, T; L^2(\Gamma))$, $w_0 \in L^2(\Omega)$ and $w_1 \in H^{-1}(\Omega)$. Then $w \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$. If $f = 0$ and $w_0 \in H_0^1(\Omega)$, $w_1 \in L^2(\Omega)$ then $w \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$.*

If $w_0 = 0$, $w_1 = 0$, $F = 0$ and $f \in L^2(0, T; L^2(\Gamma))$ then $w \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$.

In every case w depends continuously on the data in the specified spaces.

These results justify the following definition of controllability:

Definition 5 *Let $T > 0$ and*

$$\Lambda_M(T)f = w(T), \quad R_M(T) = \text{im } \Lambda_M(T) = \{w(T), \quad f \in L^2(0, T; L^2(\Gamma))\}.$$

System (1) is controllable when the map $\Lambda_M(T)$ is surjective, i.e. when $R_M(T) = L^2(\Omega)$.

The result that we shall prove is:

Theorem 6 *Let the associated wave equation be controllable at time $2T$ and let $\epsilon > 0$. Then system (1) is controllable at time $T + \epsilon$.*

Remark 7 *This observation will be important. The affine term F in (14) does depend on w_0 and w_1 because of the integration by parts in the MacCamy trick. If the given equation is (9), with $F = 0$ and $f = 0$ then formula (14) takes the form*

$$w(t) = R_+(t)w_0 + \mathcal{A}^{-1}R_-(t)w_1 + \int_0^t H(t-s) [R_+(s)w_0 + \mathcal{A}^{-1}R_-(s)w_1] \, ds \quad (16)$$

3.1 The direct inequality for Eq. (16)

Formula (4) shows a “hidden regularity” of the wave equation, and this inequality is called the “direct inequality” of the wave equation. We are going to prove an analogous result for the solution of Eq. (16), i.e. we prove:

Theorem 8 *Let $T > 0$. If $w_0 \in H_0^1(\Omega)$ and $w_1 \in L^2(\Omega)$ and let w solve Eq. (9) with $f = 0$. Then $\gamma_1 w$ belongs to $L^2(0, T; L^2(\Gamma))$ and depends continuously on w_0, w_1 , i.e. there exists M such that*

$$|\gamma_1 w|_{L^2(0, T; L^2(\Gamma))}^2 \leq M \left(|w_0|_{H_0^1(\Omega)}^2 + |w_1|_{L^2(\Omega)}^2 + |F|_{L^1(0, T; L^2(\Omega))}^2 \right). \quad (17)$$

We give the proof in the case $F = 0$ (the proof is easily adapted to $F \neq 0$). The proof uses this property, that

$$\text{if } \phi \in \text{dom } A \text{ then } \gamma_1 \phi = -D^* A \phi.$$

In order to prove Theorem 8 we introduce the notation $u(t) = R_+(t)w_0 + \mathcal{A}^{-1}R_-(t)w_1$ and

$$H_1(t) = \sum_{k=2}^{+\infty} (\mathcal{A}^{-1})^k L^{*k} = A^{-1} \sum_{k=2}^{+\infty} (\mathcal{A}^{-1})^{k-2} L^{*k}$$

so that

$$w(t) = u(t) + \mathcal{A}^{-1} \int_0^t L(t-s)u(s) \, ds + A^{-1} \int_0^t H_1(t-s)u(s) \, ds. \quad (18)$$

Then,

$$\gamma_1 u(t), \quad \gamma_1 \left(A^{-1} \int_0^t H_1(t-s)u(s) \, ds \right) = -D^* \left(\int_0^t H_1(t-s)u(s) \, ds \right)$$

are continuous functions of $w_0 \in H_0^1(\Omega)$ and $w_1 \in L^2(\Omega)$.

We study the first integral in (18),

$$\mathcal{A}^{-1} \int_0^t L(t-s)u(s) \, ds = \mathcal{A}^{-1} \int_0^t L(t-s)R_+(s)w_0 \, ds + A^{-1} \int_0^t L(t-s)R_-(s)w_1 \, ds.$$

The second term gives

$$\gamma_1 \left(A^{-1} \int_0^t L(t-s)R_-(s)w_1 \, ds \right) = -D^* \left(\int_0^t L(t-s)R_-(s)w_1 \, ds \right),$$

a continuous function of $w_1 \in L^2(\Omega)$. We study the first integral. We recall that

$$L(t)w = bR_-(t)w + \int_0^t K(t-r)R_-(r)w \, dr.$$

We consider first

$$\begin{aligned} \mathcal{A}^{-1} \int_0^t R_-(t-s)u(s) \, ds &= \\ &= \mathcal{A}^{-1} \int_0^t R_-(t-s)R_+(s)w_0 \, ds + A^{-1} \int_0^t R_-(t-s)R_-(s)w_1 \, ds. \end{aligned}$$

The trace of the second addendum is treated as above. To handle the first addendum, we use

$$R_-(\tau)R_+(r) = \frac{1}{2} (R_-(r + \tau) - R_+(r - \tau))$$

so that

$$\mathcal{A}^{-1} \int_0^t R_-(t-s)R_+(s)w_0 \, ds = \frac{1}{2}t (\mathcal{A}^{-1}R_-(t)w_0) + \frac{1}{2} \int_0^t R_+(t-2s)\mathcal{A}^{-1}w_0 \, ds.$$

The first addendum is the velocity term of the wave equation (even more regular, since $w_0 \in H_0^1(\Omega)$) and the continuity of the trace follows from the properties of the wave equation. The same property holds also for $\gamma_1 (R_+(t-2s)\mathcal{A}^{-1}w_0)$ (say on the interval $(-T, T)$).

The convolution of these terms with K retain the required properties.

4 The proof of controllability

In this section we prove Theorem 6. The proof is in two steps. In the first step we prove that $R_M(T)$ is a closed subspace of $L^2(\Omega) \times H^{-1}(\Omega)$ and that $R_M(T)^\perp$ is finite dimensional. In the second step we prove $R_M(T)^\perp = 0$, hence controllability.

4.1 The first step: $R_M(T)$ is closed and $R_M^\perp(T)$ is finite dimensional

Theorem 9 *Let the associated wave equation be controllable at time $2T$. Then $R_M(T)$ is closed with finite codimension.*

Proof. In the study of $R_M(T)$ we use the notation

$$u(t) = -\mathcal{A} \int_0^t R_-(t-s)Df(s) \, ds.$$

We fix any $\gamma < 1/4$. It is known that $\text{im } D \subseteq H^{1/2}(\Omega) \subseteq \text{dom}(-A)^\gamma$ and $(-A)^\gamma$ can be interchanged with $R_+(t)$ and $R_-(t)$ and $L(t)$.

We note that

$$\begin{aligned} \mathcal{A}^{-1} \int_0^T L(t-s)u(s) \, ds &= - \int_0^T L(t-s) \int_0^s R_-(s-r)Df(r) \, dr \, ds = \\ &= (-A)^{-\gamma} \int_0^T L(t-s) \int_0^s R_-(s-r)(-A)^\gamma Df(r) \, dr \, ds. \end{aligned}$$

This is the composition of a continuous transformation with the compact transformation $(A)^{-\gamma}$. Hence it is a compact operator. For the same, and stronger, reasons the map

$$f \mapsto K_T f = \mathcal{A}^{-1} \int_0^T L(T-s)u(s) \, ds + A^{-1} \left[\sum_{k=2}^{+\infty} (\mathcal{A}^{-1})^{k-2} L^{*k} * u \right] (T)$$

is compact, from $L^2(0, T; L^2(\Omega))$ to $L^2(\Omega)$.

Then we have

$$R_M(T) = \text{im} \left(\Lambda_0^1(T) + K_T \right) .$$

The operator $\Lambda_0^1(T)$ is surjective in $L^2(\Omega)$ by assumption while we proved that K_T is compact.

Hence, $R_M(T)$ is closed with finite codimension, as wanted. ■

4.2 The space $R_M(T)^\perp$

We characterize $R_M(T)^\perp \subseteq L^2(\Omega)$:

$$(R_M(T))^\perp = \left\{ \xi_0 \in L^2(\Omega), \quad \int_{\Omega} \xi_0(x) w(x, T) \, dx = 0 \right\} .$$

This characterization will be applied also to the elements of $R_M(T)^\perp$ and we note that

$$R_M(T + \epsilon)^\perp \subseteq R_M(T)^\perp .$$

In this computation, closure of the reachable set has no interest, so that we can work with smooth controls. For example we can assume $f \in \mathcal{D}(\Gamma \times (0, T))$.

We compute $\int_{\Omega} \xi_0(x) w(x, T) \, dx$:

$$\begin{aligned} & - \int_{\Omega} \xi_0(x) \left[\mathcal{A} \int_0^T R_-(T-s) Df(s) \, ds + \right. \\ & \left. + \mathcal{A} \int_0^T H(T-s) \int_0^s R_-(s-r) Df(r) \, dr \, ds \right] \, dx = \\ & = - \int_0^T \int_{\Gamma} f(r) D^* \mathcal{A} R_-(T-r) \xi_0 \, dr \, d\Gamma + \\ & + \int_0^T \int_{\Gamma} f(r) D^* \mathcal{A} \int_0^{T-r} H(T-r-s) R_-(s) \xi_0 \, ds \, d\Gamma \, dr = \\ & = - \int_0^T \int_{\Gamma} f(r) D^* \mathcal{A} \left[\mathcal{A}^{-1} \left(R_-(T-r) \xi_0 + \int_0^{T-r} H(T-r-s) R_-(s) \xi_0 \, ds \right) \right] \, d\Gamma \, dr . \end{aligned} \tag{19}$$

Remark 10 *Note that this is not a formal computation because the transformation $f \mapsto w$ is continuous.*

If $\xi_0 \perp R_M(T)$ then

$$D^* \mathcal{A} \left(\mathcal{A}^{-1} R_-(r) \xi_0 + \int_0^r H(r-s) \mathcal{A}^{-1} R_-(s) \xi_0 \, ds \, d\Gamma \, dt \right) = 0 \tag{20}$$

Let

$$\psi(t) = \mathcal{A}^{-1}R_-(t)\phi_1 + \int_0^t H(t-s)\mathcal{A}^{-1}R_-(s)\xi_0 \, ds.$$

We compare with (16) and we see that $\psi(t)$ solves

$$\psi'' = \Delta\psi + b\psi + \int_0^t K(t-s)\psi(s) \, ds \quad \begin{cases} \psi(0) = 0, \quad \psi'(0) = \xi_0, \\ \psi = 0 \text{ on } \partial\Omega \end{cases} \quad (21)$$

Note that $\xi_0 \in L^2(\Omega)$ so that $\psi(t) \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$.

The interpretation of (20) is simple: when ξ_0 is “smooth”, then $-D^*A = \gamma_1$ and the direct inequality shows that γ_1 is a continuous function of $\xi_0 \in L^2(\Omega)$, i.e.:

Theorem 11 *We have $\xi_0 \perp R_M(T)$ if and only if the solution of (21) has the additional property*

$$\gamma_1\psi(t) = 0 \text{ on } (0, T).$$

4.3 The proof that $R_M(T + \epsilon)^\perp = 0$

Let $\xi_0 \perp R_M(T + \epsilon)^\perp$. We are going to prove $\xi_0 = 0$. We expand

$$\xi_0(x) = \sum_{n=1}^{+\infty} \phi_n(x)\xi_n, \quad \{\xi_n\} \in l^2. \quad (22)$$

The solution ψ of system (21) has the expansion

$$\psi(x, t) = \sum_{n=1}^{+\infty} \phi_n(x)\psi_n(t)\xi_n$$

where $\psi_n(t)$ solves

$$\psi_n'' = -\lambda_n^2\psi_n + b\psi_n(t) + \int_0^t K(t-s)\psi_n(s) \, ds, \quad \psi_n(0) = 0, \quad \psi_n'(0) = 1. \quad (23)$$

The condition $\xi_0 \perp R_M(T + \epsilon)$ is the condition

$$\gamma_1\psi(t) = \sum_{n=1}^{+\infty} (\gamma_1\phi_n)\xi_n\psi_n(t) = 0, \quad 0 < t < T + \epsilon. \quad (24)$$

Remark 12 *This is a consequence of the direct inequality which implies*

$$\lim_N \sum_{n=1}^N (\gamma_1\phi_n)\xi_n\psi_n(t) = \gamma_1\psi$$

in $L^2(0, T + \epsilon; L^2(\Gamma))$. ■

The goal is the proof that equality (24) implies $\xi_0 = 0$.

In principle, it might be that the series in (24) is a finite sum, i.e. that $\xi_n = 0$ for large n .

We consider first the case that the series (22) is a finite sum and then the case that it has infinitely many nonzero elements.

The case $\xi_0 = \sum_{n=1}^N \xi_n \phi_n$ The sum cannot have only one addendum, since otherwise we should have

$$\gamma_1 \phi_{n_0} = 0 \text{ on } \Gamma$$

and ϕ_{n_0} is an eigenvector of A and Γ is the active part of $\partial\Omega$. It is known that this is not possible if there exists a time at which the wave equation is controllable. Even more, the terms with nonzero coefficients ξ_n must belong to different eigenvalues, see [4, 15].

So, the sum must have at least two terms (which correspond to different eigenvalues) and we can assume $\xi_N \neq 0$. The fact that $\xi_0 \perp R_M(T)$ implies

$$\sum_{n=1}^N \xi_n (\gamma_1 \phi_n) \psi_n(t) = 0. \quad (25)$$

Hence, also the second derivative is zero and this, coupled with (25), gives

$$\sum_{n=1}^N \lambda_n^2 \xi_n (\gamma_1 \phi_n) \psi_n(t) = 0. \quad (26)$$

We multiply (25) with λ_N^2 and we subtract from (26). We get

$$\sum_{n=1}^{N-1} (\lambda_n^2 - \lambda_N^2) \xi_n (\gamma_1 \phi_n) \psi_n(t) = 0.$$

If in this sum the nonzero coefficients $(\lambda_n^2 - \lambda_N^2) \xi_n$ correspond to the same eigenvalue, this contradicts the previous observation. But, after a finite number of iteration of the procedure surely we obtain this case, which is not possible. Hence, if $\xi_0 \neq 0$ then the sum cannot be finite.

Infinitely many nonzero entries The analysis of this case requires an intermediate step: we prove that ξ_0 is smoother then solely square integrable. In fact we prove:

Theorem 13 *Let the wave equation be controllable at time T and let $\epsilon > 0$. If $\xi_0 \in L^2(\Omega)$ belongs to $R_M(T + \epsilon)^\perp$ then we have $\xi_0 \in \text{dom } A$, i.e.*

$$\xi_0(x) = \sum_{n=1}^{+\infty} \frac{\sigma_n}{\lambda_n^2} \phi_n(x), \quad \{\sigma_n\} \in l^2.$$

We accept this theorem, whose proof is in the appendix, and we proceed to prove that $\xi_0 = 0$.

We insert the special form of $\{\xi_n\}$ in (24) and we find

$$\sum_{n=1}^{+\infty} (\gamma_1 \phi_n) \frac{\sigma_n}{\lambda_n^2} \psi_n(t) = 0.$$

The observation in Remark 12 implies that

$$\sum_{n=1}^{+\infty} (\gamma_1 \phi_n) \sigma_n \psi_n(t)$$

is convergent. And so the following equality holds:

$$\begin{aligned} 0 &= \frac{d^2}{dt^2} \sum_{n=1}^{+\infty} (\gamma_1 \phi_n) \xi_n \psi_n(t) = - \sum_{n=1}^{+\infty} (\gamma_1 \phi_n) (\lambda_n^2 \xi_n) \psi_n(t) + \\ &+ \sum_{n=1}^{+\infty} (\gamma_1 \phi_n) \left[b \psi_n(t) + \int_0^t K(t-s) \psi_n(s) ds \right] \xi_n = - \sum_{n=1}^{+\infty} (\gamma_1 \phi_n) \sigma_n \psi_n(t). \end{aligned}$$

This is the condition that

$$\xi_1 = \sum_{n=1}^{+\infty} \phi_n(x) \sigma_n = \sum_{n=1}^{+\infty} \phi_n(x) (\lambda_n^2 \xi_n) \perp R_M(T).$$

So, using $\xi_0 \perp R_M(T + \epsilon)$ we constructed a second element $\xi_1 \perp R_M(T + \epsilon)$ and the two elements ξ_0 and ξ_1 are linearly independent *thanks to the fact that (at least) two entries of ξ_0 which correspond to different eigenvalues are nonzero.*

The new element

$$\xi_1 = \sum_{n=1}^{+\infty} \phi_n(x) \sigma_n$$

has the same properties as ξ_0 and so the procedure can be repeated. We get a third element $\xi_2 \perp R_M(T + \epsilon)$,

$$\xi_2 = \sum_{n=1}^{+\infty} \phi_n(x) (\lambda_n^4 \xi_n) \in L^2(\Omega)$$

and the vectors ξ_0 , ξ_1 and ξ_2 are linearly independent *since (at least) three entries of ξ_0 which correspond to different eigenvalues are nonzero.*

The procedure can be iterated as many times as we want, because we assumed that ξ_0 has infinitely many non zero entries (while every eigenvalue has finite multiplicity) and we find that $\dim R_M(T + \epsilon)^\perp = +\infty$. We proved already that this is false and so we get that any element $\xi_0 \perp R_M(T + \epsilon)$ has to be zero: $\xi_0 = 0$. This is the result that we wanted to achieve.

5 Appendix: the proof of Theorem 13

It is known that

$$\dim \Omega = d \implies m_0 n^{2/d} \leq \lambda_n^2 \leq M n^{2/d}, \quad m_0 > 0.$$

In this proof we use the condition $\dim \Omega \leq 3$ which implies

$$\{\lambda_n^2\} \in l^2 \text{ i.e. } \sum_{n=1}^{+\infty} \frac{1}{\lambda_n^4} < +\infty \quad (27)$$

but it will be clear that this condition can be easily removed. Furthermore we present the computation in the case $b = 0$, only for simplicity of notations. We shall see that this condition has no real effect on the computations.

We use

$$\psi_n(t) = \frac{1}{\lambda_n} \xi_n \sin \lambda_n t + \int_0^t \left[\frac{1}{\lambda_n} \int_0^{t-s} K(r) \sin \lambda_n(t-s-r) \, dr \right] \psi_n(s) \, ds. \quad (28)$$

We introduce the notations

$$S_n(t) = \sin \lambda_n t, \quad C_n(t) = \cos \lambda_n t$$

and $L_n(t)$, the resolvent kernel of the bracket in (28) (with the sign changed) so that

$$\begin{aligned} L_n &= -\frac{1}{\lambda_n} K * S_n + \frac{1}{\lambda_n} (K * S_n) * L_n = \\ &= -\frac{1}{\lambda_n} K * S_n - \frac{1}{\lambda_n^2} K^{*2} * S_n^{*2} + \frac{1}{\lambda_n^2} (K^{*2} * S_n^{*2}) * L_n. \end{aligned} \quad (29)$$

The first line of (29) shows that

$$|L_n(t)| \leq M/\lambda_n \text{ for } t \in (0, T). \quad (30)$$

Due to the fact that the associated wave equation is controllable in time $2T$, hence also in larger times, we know that both $\{\Psi_n S_n\}$ and $\{\Psi_n C_n\}$ where $\Psi_n = \gamma_1 \phi_n / \lambda_n$ are Riesz sequences in $L^2(0, T; L^2(\Gamma))$ and in $L^2(0, T + \epsilon; L^2(\Gamma))$ and so the series

$$\sum_{n=1}^{+\infty} \xi_n \Psi_n S_n, \quad \sum_{n=1}^{+\infty} \xi_n \Psi_n C_n$$

converge when $\{\xi_n\} \in l^2$.

Now we use

$$\psi(x, t) = \sum_{n=1}^{+\infty} \phi_n(x) \psi_n(t) \xi_n, \quad \psi_n(t) = \frac{1}{\lambda_n} S_n(t) - \frac{1}{\lambda_n} (L_n * S_n)(t).$$

So, the condition of orthogonality to $R_M(T)$ is

$$\sum_{n=1}^{+\infty} \{\xi_n \Psi_n S_n - \xi_n \Psi_n (L_n * S_n)\} = 0.$$

This series converges and the equality holds in $L^2(0, T + \epsilon; L^2(\Gamma))$ and, as we noted, the series $\sum_{n=1}^{+\infty} \xi_n \Psi_n S_n$ converges too, so that we can write

$$\sum_{n=1}^{+\infty} \xi_n \Psi_n S_n = \sum_{n=1}^{+\infty} \xi_n \Psi_n (L_n * S_n) .$$

We prove that this function belongs to $H^1(0, T + \epsilon; L^2(\Gamma))$. We formally compute termwise the derivative of the series on the right hand side and we prove that the resulting series converges in $L^2(0, T; L^2(\Gamma))$. In fact, the derivative is

$$\begin{aligned} \sum_{n=1}^{+\infty} \Psi_n \xi_n (\lambda_n L_n * C_n) &= - \sum_{n=1}^{+\infty} \Psi_n \xi_n K * S_n * C_n - \\ &- \sum_{n=1}^{+\infty} \Psi_n \xi_n \frac{1}{\lambda_n} K^{*2} * S_n^{*2} * C_n + \\ &+ \sum_{n=1}^{+\infty} \Psi_n \xi_n \frac{1}{\lambda_n} K^{*2} * S_n^{*2} * C_n * L_n . \end{aligned} \quad (31)$$

The first and second series on the right hand side converge since

$$S_n * C_n = \frac{1}{2} t S_n , \quad S_n^{*2} * C_n = -\frac{1}{8} \left[t^2 C_n(t) - \frac{1}{\lambda_n} t S_n(t) \right] .$$

The third series converges (even uniformly) since, using (30),

$$\left| \frac{1}{\lambda_n} L_n \right| \leq \frac{M}{\lambda_n^2} . \quad (32)$$

Hence we have

$$\sum_{n=1}^{+\infty} \xi_n \Psi_n S_n \in H^1(0, T + \epsilon; L^2(\Gamma)) .$$

We combine with the fact that $\{\Psi_n S_n\}$, $\{\Psi_n C_n\}$ (and $\{\Psi_n e^{i\lambda_n t}\}$) are Riesz sequences on the *shorter* interval $(0, T)$ and we deduce (see [15, Chapt. 3])

$$\xi_n = \frac{\delta_n}{\lambda_n} , \quad \{\delta_n\} \in l^2 .$$

We replace this expression of ξ_n and we equate the derivatives of both the sides. We get:

$$\begin{aligned} \sum_{n=1}^{+\infty} \delta_n \Psi_n C_n &= \\ &- \sum_{n=1}^{+\infty} \Psi_n \frac{\delta_n}{\lambda_n} K * S_n * C_n - \\ &- \sum_{n=1}^{+\infty} \Psi_n \frac{\delta_n}{\lambda_n} \frac{1}{\lambda_n} K^{*2} * S_n^{*2} * C_n + \\ &+ \sum_{n=1}^{+\infty} \Psi_n \frac{\delta_n}{\lambda_n} \frac{1}{\lambda_n} K^{*2} * S_n^{*2} * C_n * L_n . \end{aligned}$$

Now we see that the right hand side belong to $H^1(0, T; L^2(\Gamma))$. In fact, computing the derivatives termwise of the three series we get

$$\sum_{n=1}^{+\infty} \Psi_n \delta_n K * C_n^{*2}, \quad (33)$$

$$\sum_{n=1}^{+\infty} \Psi_n \delta_n \frac{1}{\lambda_n} K^{*2} * C_n^{*2} * S_n, \quad (34)$$

$$\sum_{n=1}^{+\infty} \Psi_n \delta_n \frac{1}{\lambda_n} K^{*2} * S_n * C_n^{*2} * L_n. \quad (35)$$

The series (33) and (34) converge since

$$\begin{aligned} C_n^{*2}(t) &= \frac{1}{2} \left(t C_n(t) + \frac{1}{\lambda_n} S_n(t) \right), \\ S_n * C_n^{*2} &= \frac{1}{8} \left[\left(t^2 + \frac{1}{\lambda_n^2} \right) S_n(t) - \frac{1}{\lambda_n} t C_n(t) \right]. \end{aligned}$$

The series (34) and (35) converge, even uniformly, thanks to the inequality (32).

Hence we have

$$\sum_{n=1}^{+\infty} \delta_n \Psi_n C_n \in H^1(0, T; L^2(\Omega)) \quad \text{so that} \quad \delta_n = \frac{\sigma_n}{\lambda_n}$$

hence

$$\xi_n = \frac{\sigma_n}{\lambda_n^2},$$

as we wanted to prove.

Remark 14 *The condition $\dim \Omega \leq 3$ has been used when we replace $L_n(t)$ with its representation in the second line of (29), which has a coefficient $1/\lambda_n^2$. Then we use $\{1/\lambda_n^2\} \in l^2$. If $\dim \Omega > 3$ then we have $\{1/(\lambda_n^{2k})\} \in l^2$ provided k is sufficiently large. And we can get a factor $1/(\lambda_n^{2k})$ in (29) by taking iterates of sufficiently high order. So, the condition $\dim \Omega \leq 3$ is easily removed.*

Also the condition $b = 0$ it is easily removed: it is sufficient to replace λ_n with $\beta_n = \sqrt{\lambda_n^2 - b}$. ■

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